

A modified Gallagher's Lemma

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Abstract. First we prove a modified version of the famous Lemma on the mean square estimate for exponential sums, by plugging the Cesaro weights in the right hand side of Gallagher's inequality. Then we apply it, in order to establish a mean value estimate for the Dirichlet polynomials.

1. Introduction and statement of the results.

Gallagher's Lemma (see [Ga], Lemma 1) is a well-known general mean value estimate for series of the type

$$S(t) \stackrel{\text{def}}{=} \sum_{\nu} c(\nu) e(\nu t) ,$$

where $e(x) \stackrel{\text{def}}{=} e^{2\pi i x}$ as usual, the frequencies ν run over a (strictly increasing) sequence of real numbers and the coefficients $c(\nu)$ are complex numbers. Precisely, it states that, if $S(t)$ is absolutely convergent and $\delta \stackrel{\text{def}}{=} \theta/T$ with $\theta \in (0, 1)$, then

$$(\star) \quad \int_{-T}^T |S(t)|^2 dt \ll_{\theta} \delta^{-2} \int_{\mathbb{R}} \left| \sum_{x < \nu \leq x+\delta} c(\nu) \right|^2 dx .$$

Hereafter, according to Vinogradov's notation $A \ll_{\theta} B$ stands for $|A| \leq CB$, where $C > 0$ is an unspecified constant that depends on θ , namely $C = C(\theta)$. Typically, in the present context the bounds hold for $T \rightarrow 0$ or $T \rightarrow \infty$ (more precisely, for $|T| \leq T_0$ with a sufficiently small $T_0 > 0$ or for $T > T_0$ with a sufficiently large $T_0 > 1$).

An immediate and renowned consequence is Theorem 1 of [Ga], also known as Gallagher's Lemma for the Dirichlet series. Indeed, since an absolutely convergent Dirichlet series can be written as

$$D(t) \stackrel{\text{def}}{=} \sum_n a_n n^{it} = \sum_{\nu} c(\nu) e(\nu t)$$

by taking $\nu \stackrel{\text{def}}{=} (2\pi)^{-1} \log n$ and $c(\nu) \stackrel{\text{def}}{=} a_{e(\nu/i)}$, then, making the substitution $x = \theta \log y$ in (\star) with $\theta \stackrel{\text{def}}{=} (2\pi)^{-1}$ and recalling that $T = \theta \delta^{-1}$, one immediately has

$$(\star\star) \quad \int_{-T}^T |D(t)|^2 dt \ll T^2 \int_0^{+\infty} \left| \sum_{y < n \leq ye^{1/T}} a_n \right|^2 \frac{dy}{y} .$$

Motivated by our study on the relationship between the Selberg integral and its modification with the Cesaro weights (see [C] and [CL]), here we give a modified version of the inequality (\star) , by plugging such weights in the right hand side. Indeed, keeping Gallagher's notation, we prove the following variation of his inequality. In passing, we point out that there are further possible generalizations (see the Remark in §2).

Lemma. *Let $T > 0$ and $\delta = \theta/T$ with $\theta \in (0, 1)$. Then*

$$(\tilde{\star}) \quad \int_{-T}^T |S(t)|^2 dt \ll_{\theta} \delta^{-2} \int_{\mathbb{R}} \left| \sum_{|\nu-x| \leq \delta} (1 - |\nu-x|\delta^{-1}) c(\nu) \right|^2 dx .$$

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We remark that, if $S(t)$ is an exponential sum (in other words, if c has finite support), the integral of the upper bound in (\star) is de facto the *Selberg integral* of the arithmetic function c with a vanishing mean value in short intervals (say, c is *balanced*), i.e.

$$J_c(N, \delta) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{x < n \leq x + \delta} c(n) \right|^2,$$

where $x \sim N$ means that x is an integer of the interval $(N, 2N]$ with $N \rightarrow \infty$. In particular, when $\delta = o(N)$ and c is a real and balanced function, in [CL] (compare §4 up to page 15) it is showed that one has

$$\tilde{J}_c(N, \delta) \stackrel{\text{def}}{=} \sum_{x \sim N} \left| \sum_{0 \leq |n-x| \leq \delta} \left(1 - \frac{|n-x|}{\delta}\right) c(n) \right|^2 \ll J_c(N, \delta) + \delta^3 \|c\|_\infty^2,$$

where $\|c\|_\infty \stackrel{\text{def}}{=} \max_{N-\delta < n \leq 2N+\delta} |c(n)|$ and $\tilde{J}_c(N, \delta)$ is the *modified* Selberg integral of c , that apparently emulates the integral in the right hand side of (\star) . From this point of view our Lemma can be proposed as a sort of refinement of Gallagher's one. In fact, the above inequality suggests that the modified Selberg integral should be easier to study than the corresponding Selberg integral. Actually, we expect this to be true, due to the further averaging over the inner short sum coming from Cesaro weights (compare the discussion in §0 of [CL] after Theorem 2).

Noteworthily, with the aid of our Lemma the first author [C] has recently derived a non trivial estimate for the Selberg integral of the three-divisor function d_3 under a Conjecture, on the corresponding modified Selberg integral, that is analyzed in [CL].

Here we also obtain a non trivial consequence for Dirichlet polynomials, though it is not as immediate as $(\star\star)$ whenever (\star) is given.

Theorem. *For every Dirichlet polynomial $D(t) = \sum_n a_n n^{it}$ one has, for $T \rightarrow \infty$,*

$$(\star\star) \quad \int_{-T}^T |D(t)|^2 dt \ll T^2 \int_1^{+\infty} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - \frac{|n-y|}{y/T}\right) a_n \right|^2 \frac{dy}{y} + \int_1^{+\infty} \left(\sum_{y-\Delta \leq n \leq y+\Delta} |a_n| \right)^2 \frac{dy}{y},$$

where $\Delta = \Delta(y, T) \stackrel{\text{def}}{=} y/T + O(y/T^2)$.

We think that such a result may be easily generalized to any absolutely convergent series.

An application of our Theorem concerns the special case of Dirichlet polynomials approximating Dirichlet series on the critical line $1/2 + it$, by taking $a_n = w(n)b(n)n^{-1/2}$ with a bounded weight $w(n)$ and an “essentially bounded” arithmetic function b . Indeed, we prove the following consequence of our Theorem.

Corollary. *Let us consider $P(t) \stackrel{\text{def}}{=} \sum_{N_1 \leq n \leq N_2} \frac{w(n)b(n)}{n^{1/2+it}}$, where N_1, N_2 are positive integers, w is uniformly bounded and supported in $[N_1, N_2]$, and $|b(n)| \ll_\varepsilon n^\varepsilon$, $\forall \varepsilon > 0$. Then, as $T \rightarrow \infty$, we have*

$$\int_{-T}^T |P(t)|^2 dt \ll_\varepsilon T^2 \int_{N_1/2}^{3N_2/2} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - \frac{|n-y|}{y/T}\right) \frac{w(n)b(n)}{n^{1/2}} \right|^2 \frac{dy}{y} + \frac{N_2^{1+\varepsilon}}{T^2}.$$

At least in principle, this Corollary may be applied within the specific situation given in [C0] for the $2k$ -th moments of the Riemann ζ function on the critical line, i.e.

$$I_k(T) \stackrel{\text{def}}{=} \int_T^{2T} \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} dt.$$

Indeed, Theorem 1.1 of [C0] links $I_k(T)$ to $\tilde{J}_k(N, \delta)$, the Selberg integral of the k -divisor function¹

$$d_k(n) \stackrel{def}{=} \sum_{\substack{n_1, \dots, n_k \\ n_1 \cdots n_k = n}} 1, \quad (k \in \mathbb{N}).$$

By following [C0] approach, in a forthcoming paper it will be exploited the same link with $\tilde{J}_k(N, \delta)$, where essentially it replaces the corresponding Selberg integral in the upper bound for $I_k(T)$, so that our Corollary may enter the scene for $b(n) = \tilde{d}_k(n)$, the balanced part of $d_k(n)$ (compare §3 of [CL]). In particular, assuming that the Conjecture given in [CL] holds for the modified Selberg integral of d_3 , such a new approach would lead to the so-called “weak sixth moment for the Riemann zeta-function” (see [CL], §8, 9).

2. Proof of the results.

PROOF OF THE LEMMA. By following [Ga], where it is introduced the auxiliary function

$$C_\delta(x) \stackrel{def}{=} \sum_{\nu} c(\nu) F_\delta(x - \nu),$$

with $F_\delta(y) = \delta^{-1}$ or 0 according as $|y| \leq \delta/2$ or not, analogously we write

$$\delta^{-2} \int_{\mathbb{R}} \left| \sum_{|\nu-x| \leq \delta} (1 - |\nu-x|\delta^{-1}) c(\nu) \right|^2 dx = \int_{\mathbb{R}} |\tilde{C}_\delta(x)|^2 dx,$$

where $\tilde{C}_\delta(x) \stackrel{def}{=} \sum_{\nu} c(\nu) \tilde{F}_\delta(x - \nu)$ with $\tilde{F}_\delta(y) \stackrel{def}{=} \max(\delta^{-1} - |y|\delta^{-2}, 0)$. The Fourier transform of $\tilde{C}_\delta(x)$ is

$$\begin{aligned} \hat{\tilde{C}}_\delta(y) &\stackrel{def}{=} \int_{\mathbb{R}} \tilde{C}_\delta(x) e(-xy) dx = \sum_{\nu} c(\nu) \int_{\mathbb{R}} \tilde{F}_\delta(x - \nu) e(-xy) dx = \\ &= \sum_{\nu} c(\nu) e(-\nu y) \int_{\mathbb{R}} \tilde{F}_\delta(t) e(-ty) dt = \hat{\tilde{F}}_\delta(y) S(-y). \end{aligned}$$

Thus, by Plancherel's theorem one has

$$\int_{\mathbb{R}} |\tilde{C}_\delta(x)|^2 dx = \int_{\mathbb{R}} |\hat{\tilde{C}}_\delta(y)|^2 dy = \int_{\mathbb{R}} |\hat{\tilde{F}}_\delta(y) S(-y)|^2 dy,$$

where the Fourier transform of $\tilde{F}_\delta(x)$ is given by²

$$\hat{\tilde{F}}_\delta(y) = \delta^{-2} \widehat{\max}(\delta - |y|, 0) = \delta^{-2} \frac{\sin^2(\pi \delta y)}{\pi^2 y^2} = \left(\frac{\sin(\pi \delta y)}{\pi \delta y} \right)^2 = \hat{F}_\delta(y)^2.$$

Since $\frac{\sin(\pi \delta y)}{\pi \delta y} \gg 1$ when $|y| \leq T$, with a constant depending on θ (recall $\delta = \theta/T$), then

$$\int_{-T}^T |S(t)|^2 dt = \int_{-T}^T |S(-t)|^2 dt \ll_{\theta} \int_{\mathbb{R}} |\hat{\tilde{F}}_\delta(y) S(-y)|^2 dy = \delta^{-2} \int_{\mathbb{R}} \left| \sum_{|\nu-x| \leq \delta} (1 - |\nu-x|\delta^{-1}) c(\nu) \right|^2 dx$$

¹ Namely, $d_k(n)$ is the number of ways to write $n \geq 1$ as a product of k positive integers and the Dirichlet series generating $d_k(n)$ is the same $\zeta^k(s)$.

² Note that, as in Gallagher's, we have a normalization condition even in our case, i.e.

$$\int_{\mathbb{R}} \tilde{F}_\delta(y) dy = \int_{\mathbb{R}} F_\delta(y) dy = 1.$$

and the Lemma is proved. \square

Remark. Evidently, any power of the original transform $\widehat{F}_\delta(y)$ is $\gg 1$ (with the implicit constant still depending on θ). However, apart from our case of the square that is linked to the Cesaro weights, higher powers would lead to different and much more complicated weights. Nevertheless, we are going to explore such implications further in the future.

PROOF OF THE THEOREM. Let us apply our Lemma to

$$D(t) = \sum_{\nu} c(\nu) e(\nu t), \text{ with } \nu = (2\pi)^{-1} \log n, \quad c(\nu) = a_{e(\nu/i)},$$

by taking $\theta = (2\pi)^{-1}$, $T = \theta\delta^{-1}$ and $x = \theta \log y$. Thus, we get

$$\begin{aligned} \int_{-T}^T |D(t)|^2 dt &\ll \delta^{-2} \int_{\mathbb{R}} \left| \sum_{|\nu-x| \leq \delta} (1 - |\nu-x|\delta^{-1}) c(\nu) \right|^2 dx \ll \\ &\ll T^2 \int_0^{+\infty} \left| \sum_{|\log n - \log y| \leq 1/T} (1 - T|\log n - \log y|) a_n \right|^2 \frac{dy}{y} \ll \\ &\ll T^2 \int_1^{+\infty} \left| \sum_{y-(1-1/\tau)y \leq n \leq y+(\tau-1)y} \left(1 - T \left| \log \left(1 + \frac{n-y}{y}\right) \right| \right) a_n \right|^2 \frac{dy}{y}, \end{aligned}$$

where we have set $\tau \stackrel{\text{def}}{=} e^{1/T} > 1$ (see that $\tau \rightarrow 1$ as $T \rightarrow \infty$, so the n -sum is empty for $0 < y < 1$).

Since Taylor expansion yields

$$y - (1 - 1/\tau)y = y - \frac{y}{T} + O\left(\frac{y}{T^2}\right), \quad y + (\tau - 1)y = y + \frac{y}{T} + O\left(\frac{y}{T^2}\right),$$

then the Cesaro weight, $1 - T|\log(1 + \frac{n-y}{y})|$, is bounded for the present range of n , while in both ranges

$$0 \leq |n - (y - y/T)| \ll y/T^2 \text{ and } 0 \leq |n - (y + y/T)| \ll y/T^2$$

we have

$$1 - T \left| \log \left(1 + \frac{n-y}{y}\right) \right| \ll \frac{1}{T}.$$

Accordingly we write

$$\begin{aligned} \int_{-T}^T |D(t)|^2 dt &\ll T^2 \int_1^{+\infty} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - T \left| \log \left(1 + \frac{n-y}{y}\right) \right| \right) a_n \right|^2 \frac{dy}{y} + \\ &+ \int_1^{+\infty} \left(\sum_{0 \leq |n-(y-y/T)| \ll y/T^2} |a_n| + \sum_{0 \leq |n-(y+y/T)| \ll y/T^2} |a_n| \right)^2 \frac{dy}{y}. \end{aligned}$$

Then $(\star\star)$ follows since by Taylor expansion again we have

$$T \left| \log \left(1 + \frac{n-y}{y}\right) \right| - \frac{|n-y|}{y/T} \ll \frac{T(n-y)^2}{y^2} \ll \frac{1}{T}$$

for $y - y/T \leq n \leq y + y/T$ and this yields

$$T^2 \int_1^{+\infty} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - T \left| \log \left(1 + \frac{n-y}{y}\right) \right| \right) a_n \right|^2 \frac{dy}{y} \ll$$

$$\ll T^2 \int_1^{+\infty} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - \frac{|n-y|}{y/T}\right) a_n \right|^2 \frac{dy}{y} + \int_1^{+\infty} \left(\sum_{-\Delta \leq n-y \leq \Delta} |a_n| \right)^2 \frac{dy}{y},$$

where recall that $\Delta = y/T + O(y/T^2)$. □

PROOF OF THE COROLLARY. Let us apply the Theorem to $D(-t)$ with $a_n = w(n)b(n)n^{-1/2}$ and write

$$\int_{-T}^T |P(t)|^2 dt \ll_\varepsilon T^2 \int_1^{+\infty} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - \frac{|n-y|}{y/T}\right) \frac{w(n)b(n)}{n^{1/2}} \right|^2 \frac{dy}{y} + N_2^\varepsilon \int_1^{+\infty} \left(\sum_{y-\Delta \leq n \leq y+\Delta} \frac{1}{\sqrt{n}} \right)^2 \frac{dy}{y}.$$

Since w has support in $[N_1, N_2]$, then

$$\begin{aligned} \int_{-T}^T |P(t)|^2 dt &\ll_\varepsilon T^2 \int_{N_1/2}^{3N_2/2} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - \frac{|n-y|}{y/T}\right) \frac{w(n)b(n)}{n^{1/2}} \right|^2 \frac{dy}{y} + N_2^\varepsilon \int_{N_1/2}^{3N_2/2} \frac{\Delta(y, T)^2}{y^2} dy \ll_\varepsilon \\ &\ll_\varepsilon T^2 \int_{N_1/2}^{3N_2/2} \left| \sum_{y-y/T \leq n \leq y+y/T} \left(1 - \frac{|n-y|}{y/T}\right) \frac{w(n)b(n)}{n^{1/2}} \right|^2 \frac{dy}{y} + N_2^{1+\varepsilon} \frac{1}{T^2}, \end{aligned}$$

using $\Delta = \Delta(y, T) \ll y/T$, whence the Corollary. □

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REFERENCES

- [C0] Coppola, G. - *On the Selberg integral of the k -divisor function and the $2k$ -th moment of the Riemann zeta-function* - Publ. Inst. Math. (Beograd) (N.S.) **88(102)** (2010), 99–110. [MR 2011m:11173](#) - available online
- [C] Coppola, G. - *On the Selberg integral of the three-divisor function d_3* - available online at the address <http://arxiv.org/abs/1207.0902> (see version 3)
- [CL] Coppola, G. and Laporta, M. - *Generations of correlation averages* - <http://arxiv.org/abs/1205.1706> (see version 3)
- [Ga] Gallagher, P. X. - *A large sieve density estimate near $\sigma = 1$* - Invent. Math. **11** (1970), 329–339. [MR 43#4775](#)

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